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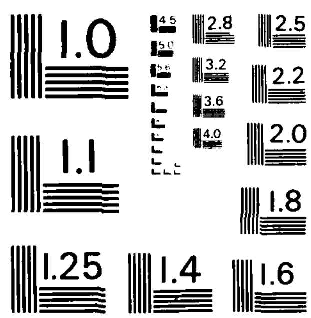
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In this paper ~~we study~~ the one-dimensional on-line bin packing problem. A list of pieces, each of size between zero and unity are to be packed, in order of their arrival, into a minimum number of unit-capacity bins. ~~We~~ present a new linear-time algorithm, the Modified Harmonic Algorithm, and show that it has an asymptotic worst-case performance ratio less that $\frac{3}{2}+\frac{1}{9}+\frac{1}{222}=1.61(561)^*$. The analysis of the algorithm's performance involves a novel use of weighting functions. In addition, we also show that for a large class of linear-time on-line algorithms,

the performance ratio is at least $\frac{3}{2} + \frac{1}{9} = 1.61^*$.

On-line Bin Packing in Linear Time

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Abstract. In this paper we study the one-dimensional on-line bin packing problem. A list of pieces, each of size between zero and unity are to be packed, in order of their arrival, into a minimum number of unit-capacity bins. We present a new linear-time algorithm, the Modified Harmonic Algorithm, and show that it has an asymptotic worst-case performance ratio less than $\frac{3}{2} + \frac{1}{9} + \frac{1}{222} = 1.61(561)^*$. The analysis of the algorithm's performance involves a novel use of weighting functions. We also show that for a large class of linear-time on-line algorithms, the performance ratio is at least $\frac{3}{2} + \frac{1}{9} = 1.61^*$.

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1. Introduction

Let $L = (p_1, p_2, \ldots, p_n)$ be a list of pieces with sizes in the interval (0, 1]. The one-dimensional *bin packing problem* is to pack the pieces into a minimum number of bins in such a way that the sum of the piece sizes in each bin is at most one. As this problem is known to be NP-complete [GJ79, K72], much work has been done in the study of approximation algorithms; a survey of these results is given in [CGJ83].

For any (heuristic) bin packing algorithm A, let $A(L)$ denote the number of bins used by algorithm A in packing list L, and let $OPT(L)$ denote the minimum (optimum) number of bins required to pack list L. We are concerned with the asymptotic worst-case *performance ratio*

$$R_A = \lim_{n \to \infty} \max_{OPT(L)=n} \frac{A(L)}{OPT(L)}.$$

Thus, we would like to construct an algorithm A which has a performance ratio close to one. Intuitively, we want an algorithm that minimizes, for large lists, the worst-case percentage of excess bins used compared to an optimal packing.

In this paper we concern ourselves with algorithms for which the pieces in list L are available one at a time, and each piece must be packed in some bin before the next piece is available; such an algorithm is referred to as *on-line*. Previously known on-line algorithms include the $O(n)$ Next-Fit (NF), and the $O(n \log n)$ First-Fit (FF) [J73, J74, JDUGG74], the $O(n)$ Harmonic (H) [LL83], the $O(n \log n)$ Refined First-Fit (RFF) [Y80], and the $O(n \log n)$ Doubly-Refined First-Fit ($DRFF$) [B79b]. These algorithms have the following performance ratios: $R_{NF} = 2$, $R_{FF} = 1.7$, $R_H = 1.692\ldots$, $R_{RFF} = 1.6^*$, and $R_{DRFF} < 1.64$. In this paper we present a new linear-time algorithm which we call Modified Harmonic (MH) and show that $R_{MH} < 1.61(561)^*$.

On the lower bound side, Yao [Y80] showed that for any on-line algorithm, the performance ratio is at least 1.5. This lower bound was further improved to 1.536... independently by Brown [B79a] and Liang [L80].

It should be observed that considerably better performance ratios exist for algorithms which are not on-line. For instance, running the First-Fit Algorithm on pieces that have been ordered by decreasing size gives the First-Fit Decreasing (FFD) Algorithm [J73, J74, JDUGG74], for which $R_{FFD} = \frac{11}{9} = 1.2^*$.

Friesen and Langston [FrL81] devised a hybrid algorithm with a performance ratio of 1.2, and Garey and Johnson [GJ81] modified First-Fit Decreasing to obtain an algorithm (MFFD) with $R_{MFFD} = \frac{71}{60} = 1.183^*$. Fernandez de la Vega and Lueker [FeL81] showed that for every $\epsilon > 0$, there is a linear-time algorithm $A[\epsilon]$ with $R_{A[\epsilon]} \leq 1 + \epsilon$. More recently, Karmarkar and Karp [KK82] presented an algorithm that is asymptotically optimal; i.e., has performance ratio one.

In Section 2, we present our Modified Harmonic Algorithm, and describe the packings produced. In Section 3, we use a novel weighting function scheme to analyze the algorithm. In Section 4, we characterize a large class of linear-time on-line algorithms for which the performance ratio is at least $\frac{3}{2} + \frac{1}{9} = 1.61^*$, suggesting that it may be difficult to improve on our Modified Harmonic Algorithm if we restrict ourselves to linear time. In Section 5, we summarize our results and make some further observations.

2. The Modified Harmonic Algorithm

The Modified Harmonic Algorithm (MH) is based on three previously known on-line algorithms: the Refined First-Fit Algorithm of Yao [Y80], the Next-Fit Algorithm of Johnson [J73], and the Harmonic Algorithm of Lee and Lee [LL83]. Because the latter two algorithms are needed to describe our algorithm, we first briefly describe them.

The Next-Fit Algorithm operates as follows. Initially, the empty bins are indexed as bin_1, bin_2, ... Piece p_1 is packed in bin_1. Suppose that $p_1, p_2, ..., p_{i-1}$ have been packed, and p_i is the next piece to be packed. Let j be the largest index such that bin_j is nonempty. If p_i will fit in bin_j, then p_i is packed in bin_j; otherwise p_i is packed in bin_{j+1}.

The Harmonic Algorithm is based on the harmonic partition of the interval (0, 1]:

$$(0, 1] = \bigcup_{j=1}^{k} I_j, \text{ where } I_j = (1/(j+1), 1/j], \ 1 \leq j < k, \text{ and } I_k = (0, 1/k],$$

for some $k > 1$. A piece p is called an I_j-piece if $p \in I_j$, $1 \leq j \leq k$. Initially, the set of empty bins is divided into k infinite classes: bins of type B_j, $1 \leq j \leq k$. A bin of type B_j is used to pack only I_j-pieces. Note that j I_j-pieces can be packed in a bin of type B_j for $1 \leq j < k$. Suppose that

$p_1, p_2, \ldots, p_{i-1}$ have been packed, and p_i is the next piece to be packed. If p_i is an I_j-piece for some j, $1 \leq j < k$, then it is packed in a nonempty bin of type B_j that contains fewer than j pieces, if one exists; if no such bin exists, then p_i is packed in an empty bin of type B_j. If p_i is an I_k-piece, then it is packed in bins of type B_k by Next-Fit.

Now we are ready to describe the Modified Harmonic Algorithm. The algorithm is based on the following partition of the interval (0, 1]:

$$(0, 1] = \bar{I}_1 \bigcup \bar{I}_2 \bigcup_{j=1}^{k} I_j,$$

where $\bar{I}_1 = (1 - y, 1]$, $\bar{I}_2 = (y, 1/2]$, $I_1 = (1/2, 1 - y]$, $I_2 = (1/3, y]$,

$I_j = (1/(j+1), 1/j]$, $3 \leq j < k$, and $I_k = (0, 1/k]$,

for some y and k, $1/3 < y < 1/2$ and $k \geq 3$ (exact values for y and k will be given in Section 3). A piece p is called an

$\bar{I}_1$-piece if $p \in \bar{I}_1$,

$\bar{I}_2$-piece if $p \in \bar{I}_2$,

I_j-piece if $p \in I_j$ for some j, $1 \leq j \leq k$.

Initially, the set of empty bins is divided into $k + 2$ infinite classes: bins of type $\bar{B}_1$, $\bar{B}_2$, and B_j, $1 \leq j \leq k$. A bin of type $\bar{B}_1$ is used to pack only $\bar{I}_1$-pieces, a bin of type $\bar{B}_2$ is used to pack only $\bar{I}_2$-pieces, and a bin of type B_j, $2 \leq j \leq k$, is used to pack only I_j-pieces. All I_1-pieces are packed in bins of type B_1. In addition, some of the I_ρ-pieces for $2 \leq \rho \leq 3$ or $6 \leq \rho \leq k - 2$, are also packed in bins of type B_1. In particular, for each ρ, a fixed fraction $\frac{1}{m_\rho}$ (values for the m_ρ's will be given in Section 3) of the I_ρ-pieces are packed in bins of type B_1; if there are a sufficient number of I_1-pieces, then each bin of type B_1 will also contain an I_1-piece. Thus, each nonempty bin of type B_1 will contain an I_1-piece and/or I_ρ-pieces (at most $\max(1, \lfloor \rho y \rfloor)$ of them) for one ρ, $2 \leq \rho \leq 3$ or $6 \leq \rho \leq k - 2$. The algorithm packs a list in such a way that, at any stage in the packing:

(1) each nonempty bin of type $\bar{B}_1$ contains 1 $\bar{I}_1$-piece,

(2) each nonempty bin of type $\bar{B}_2$ (except possibly the last one) contains 2 $\bar{I}_2$-pieces,

(3) each nonempty bin of type B_j, $2 \leq j < k$, (except possibly the last one) contains j I_j-pieces,

(4) each nonempty bin of type B_k (except possibly the last one) is at least $\frac{k-1}{k}$ full,

and

(5) each nonempty bin of type B_1

(i) contains only an I_1-piece,

(ii) contains only an I_2-piece or an I_3-piece,

(iii) contains an I_1-piece together with an I_2-piece or an I_3-piece,

(iv) contains $\lfloor \rho y \rfloor$ I_ρ-pieces for some ρ, $6 \leq \rho \leq k-2$,

(v) contains an I_1-piece, and $\lfloor \rho y \rfloor$ I_ρ-pieces for some ρ, $6 \leq \rho \leq k-2$,

(vi) contains at least one, and at most $\lfloor \rho y \rfloor - 1$ I_ρ-pieces for some ρ, $6 \leq \rho \leq k-2$,

or

(vii) contains an I_1-piece together with at least one, and at most $\lfloor \rho y \rfloor - 1$ I_ρ-pieces for some ρ, $6 \leq \rho \leq k-2$.

Moreover, if there is a bin as in (i), then there can be no bins as in (ii), (iv), or (vi). Also, for each ρ, $6 \leq \rho \leq k-2$, the number of bins as in (vi) plus the number of bins as in (vii) is at most one.

When we say *harmonic pack* (p_i, I_j), where p_i is an I_j-piece, $2 \leq j < k$, we mean:

if there exists a nonempty bin of type B_j containing fewer than j I_j-pieces

then pack p_i in that bin

else pack p_i in an empty bin of type B_j.

We now give a precise statement of our algorithm.

Modified Harmonic Algorithm

for $i := 1$ *to* n *do*

begin

case p_i *in*

$\bar{I}_1$: pack p_i in an empty bin of type $\bar{B}_1$

$\bar{I}_2$: *if* there exists a bin of type $\bar{B}_2$ containing only one $\bar{I}_2$-piece

then pack p_i in that bin

else pack p_i in an empty bin of type $\bar{B}_2$

I_j $(j = 4, 5,$ or $k-1)$: *harmonic pack* (p_i, I_j)

I_k : pack p_i in bins of type B_k by Next-Fit

I_1 : *if* there exists a nonempty bin of type B_1 that does not contain an I_1-piece

then pack p_i in that bin

else pack p_i in an empty bin of type B_1

I_ρ $(2 \leq \rho \leq 3)$: *if* p_i is the $(m_\rho r)^{th}$ I_ρ-piece to arrive thus far for some integer $r \geq 1$

then *if* there exists a bin of type B_1 containing only an I_1-piece

then pack p_i in that bin

else pack p_i in an empty bin of type B_1

else *harmonic pack* (p_i, I_ρ)

I_ρ $(6 \leq \rho \leq k-2)$: *if* p_i is the $(m_\rho r)^{th}$ I_ρ-piece to arrive thus far for some integer $r \geq 1$

then *if* there exists a bin of type B_1 containing at least one, and at most

$\lfloor \rho y \rfloor - 1$ I_ρ-pieces

then pack p_i in that bin

else *if* there exists a bin of type B_1 containing only an I_1-piece

then pack p_i in that bin

else pack p_i in an empty bin of type B_1

else *harmonic pack* (p_i, I_ρ)

end.

It is easy to see that the Modified Harmonic Algorithm runs in linear time and uses linear space. Note that in the above algorithm, we had implicitly assumed that the m_ρ's are integers. The values we will be specifying in Section 3 will not be integers. In this case we require that at any stage if a_ρ is the number of I_ρ-pieces that have been packed, then $\left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor$ of them have been packed in bins of type B_1. It is easy to modify the algorithm accordingly.

3. Analysis of the Algorithm

In this section, we use weighting functions to analyze the performance of the Modified Harmonic Algorithm. Throughout this section we shall be considering only the nonempty, i.e. packed, bins. The weight of a bin is defined to be the sum of the weights of all the pieces in the bin. We shall assign weights to pieces in such a way that the average weight of all but a constant number of bins packed by our algorithm is at least one.

Let $\overline{w}_1$ and $\overline{w}_2$ be the weights of an $\overline{I}_1$-piece and an $\overline{I}_2$-piece, respectively. Let w_j be the weight of an I_j-piece, $1 \leq j < k$, and let $w_k(p)$ be the weight of an I_k-piece p.

Since a bin can contain one $\overline{I}_1$-piece, two $\overline{I}_2$-pieces, four I_4-pieces, five I_5-pieces, or $k-1$ I_{k-1}-pieces, we have

$$\overline{w}_1 = 1,$$

$$\overline{w}_2 = \frac{1}{2},$$

$$w_4 = \frac{1}{4},$$

$$w_5 = \frac{1}{5},$$

$$w_{k-1} = \frac{1}{k-1}.$$

Since a bin of type B_k will be at least $\frac{k-1}{k}$ full, we assign to an I_k-piece p the weight

$$w_k(p) = \frac{k}{k-1} p.$$

Interval containing piece s	weight of piece s for	
	$\alpha > 0$	$\alpha = 0$
$\bar{I}_1$: $(1 - y, 1]$	1	1
I_1 : $(1/2, 1 - y]$	1	0
$\bar{I}_2$: $(y, 1/2]$	$\frac{1}{2}$	$\frac{1}{2}$
I_2 : $(1/3, y]$	$\frac{1}{2} - \frac{1}{2m_2}$	$\frac{1}{2} + \frac{1}{2m_2}$
I_3 : $(1/4, 1/3]$	$\frac{1}{3} - \frac{1}{3m_3}$	$\frac{1}{3} + \frac{2}{3m_3}$
I_4 : $(1/5, 1/4]$	$\frac{1}{4}$	$\frac{1}{4}$
I_5 : $(1/6, 1/5]$	$\frac{1}{5}$	$\frac{1}{5}$
I_6 : $(1/7, 1/6]$	$\frac{1}{6} - \frac{1}{6m_6}$	$\frac{1}{6} - \frac{1}{6m_6} + \frac{1}{m_6\lfloor 6y \rfloor}$
$\vdots$	$\vdots$	$\vdots$
I_ρ : $(1/(\rho + 1), 1/\rho]$	$\frac{1}{\rho} - \frac{1}{\rho m_\rho}$	$\frac{1}{\rho} - \frac{1}{\rho m_\rho} + \frac{1}{m_\rho \lfloor \rho y \rfloor}$
$\vdots$	$\vdots$	$\vdots$
I_{k-2} : $(1/(k-1), 1/(k-2)]$	$\frac{1}{k-2} - \frac{1}{(k-2)m_{k-2}}$	$\frac{1}{k-2} - \frac{1}{(k-2)m_{k-2}} + \frac{1}{m_{k-2}\lfloor (k-2)y \rfloor}$
I_{k-1} : $(1/k, 1/(k-1)]$	$\frac{1}{k-1}$	$\frac{1}{k-1}$
I_k : $(0, 1/k]$	$\frac{k}{k-1} s$	$\frac{k}{k-1} s$

Table I. Weighting functions for $\alpha > 0$ and $\alpha = 0$.

Recall that, for $2 \leq \rho \leq 3$ or $6 \leq \rho \leq k - 2$, some of the I_ρ-pieces are packed in bins of type B_1. The weights we assign to the I_1-pieces and I_ρ-pieces depend on the input list L. Let α be the number of bins in the packing of L produced by the Modified Harmonic Algorithm that contain only an I_1-piece. The weights we assign to the I_1-pieces and I_ρ-pieces depend on whether $\alpha > 0$ or $\alpha = 0$.

For $\alpha > 0$, there are bins containing only an I_1-piece; moreover, every bin of type B_1 contains an I_1-piece. We assign weights to the I_1-pieces and I_ρ-pieces in such a way that the average weight of all bins of types B_1 and B_ρ is one:

$$w_1 = 1,$$

$$w_\rho = \frac{1}{\rho}\left(1 - \frac{1}{m_\rho}\right) \text{ for } 2 \leq \rho \leq 3 \text{ or } 6 \leq \rho \leq k - 2.$$

For $\alpha = 0$, not all the bins of type B_1 contain an I_1-piece. Again, we assign weights to the I_1-pieces and I_ρ-pieces in such a way that the average weight of all bins of types B_1 and B_ρ is one:

$$w_1 = 0,$$

$$w_\rho = \frac{1}{\rho}\left(1 - \frac{1}{m_\rho}\right) + \frac{1}{m_\rho} \text{ for } 2 \leq \rho \leq 3,$$

$$w_\rho = \frac{1}{\rho}\left(1 - \frac{1}{m_\rho}\right) + \frac{1}{m_\rho \lfloor \rho y \rfloor} \text{ for } 6 \leq \rho \leq k - 2.$$

Table I summarizes the above weighting functions.

We make use of these weighting functions to analyze our algorithm's performance. Letting $W(L)$ be the sum of the weights of all the pieces in list L, we show in Lemma 1 that

$$MH(L) < W(L) + 2k - 7,$$

and show in Lemma 2 that

$$W(L) < \left(\frac{3}{2} + \frac{1}{9} + \frac{1}{222}\right) OPT(L).$$

Combining these results gives us the following bound on our algorithm's performance:

$$R_{MH} < \frac{3}{2} + \frac{1}{9} + \frac{1}{222}.$$

In Lemma 3, we prove that this bound is essentially tight. The results of these three lemmas are combined to give Theorem 1.

Lemma 1. For any list L, $MH(L) < W(L) + 2k - 7$.

Proof. Let $\bar{b}_1$ and $\bar{b}_2$ be the number of bins of type $\bar{B}_1$ and $\bar{B}_2$, respectively, and let b_j be the number of bins of type B_j, $1 \leq j \leq k$. It is clear that

$$MH(L) = \bar{b}_1 + \bar{b}_2 + \sum_{j=1}^{k} b_j.$$

Let $\bar{W}_1$ and $\bar{W}_2$ be the sum of the weights of all the $\bar{I}_1$-pieces and $\bar{I}_2$-pieces, respectively, and let W_j be the sum of the weights of all the I_j-pieces, $1 \leq j \leq k$. It is clear that

$$W(L) = \bar{W}_1 + \bar{W}_2 + \sum_{j=1}^{k} W_j.$$

It is easy to see that we have constructed the weighting functions in such a way that each bin (except possibly the last one) of type $\bar{B}_1$, $\bar{B}_2$, B_4, B_5, or B_{k-1}, has weight precisely one. So

$$\bar{b}_1 = \bar{W}_1,$$

$$\bar{b}_2 < \bar{W}_2 + 1,$$

$$b_4 < W_4 + 1,$$

$$b_5 < W_5 + 1,$$

$$b_{k-1} < W_{k-1} + 1.$$

Since each bin of type B_k (except possibly the last one) must be at least $\frac{k-1}{k}$ full, its total weight is at least one, and so

$$b_k < W_k + 1.$$

Let a_j be the number of I_j-pieces, $1 \leq j \leq k$. To analyze the weights of bins of types B_1 and B_ρ, $2 \leq \rho \leq 3$ or $6 \leq \rho \leq k - 2$, we consider separately the two weighting functions, and show that the lemma holds both for $\alpha > 0$ and $\alpha = 0$.

Case 1. $\alpha > 0$. Every bin of type B_1 has an I_1-piece, and so

$$b_1 = W_1.$$

For $2 \leq \rho \leq 3$ or $6 \leq \rho \leq k - 2$, not all of the I_ρ-pieces are packed in bins of type B_ρ. In particular, we have

$$b_\rho = \left\lceil \frac{1}{\rho}\left(a_\rho - \left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor\right) \right\rceil$$
$$< a_\rho\left(\frac{1}{\rho} - \frac{1}{\rho m_\rho}\right) + 1$$
$$= W_\rho + 1.$$

Summing all the inequalities for Case 1, we get

$$MH(L) < W(L) + k.$$

Case 2. $\alpha = 0$. A bin of type B_1 containing an I_ρ-piece for some ρ, $2 \le \rho \le 3$ or $6 \le \rho \le k - 2$, is called a bin of type $B_{1,\rho}$. Since $\alpha = 0$, every bin of type B_1 is of one of the $B_{1,\rho}$ types. Let $b_{1,\rho}$ be the number of bins of type $B_{1,\rho}$. Clearly $b_1 = \sum_\rho b_{1,\rho}$. Noting that for $\alpha = 0$ the weight of an I_1-piece is 0, the weights of the I_ρ-pieces must "compensate". For $2 \le \rho \le 3$, we have

$$b_\rho + b_{1,\rho} = \left\lceil \frac{1}{\rho}\left(a_\rho - \left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor\right) \right\rceil + \left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor$$
$$< \frac{a_\rho}{\rho}\left(1 - \frac{1}{m_\rho}\right) + \frac{a_\rho}{m_\rho} + 1$$
$$= a_\rho\left(\frac{1}{\rho} + \frac{\rho - 1}{\rho m_\rho}\right) + 1$$
$$= W_\rho + 1.$$

For $6 \le \rho \le k - 2$, each bin (except possibly the last one) of type $B_{1,\rho}$ contains $\lfloor \rho y \rfloor$ I_ρ-pieces. As above, we have

$$b_\rho + b_{1,\rho} = \left\lceil \frac{1}{\rho}\left(a_\rho - \left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor\right) \right\rceil + \left\lceil \frac{1}{\lfloor \rho y \rfloor} \left\lfloor \frac{a_\rho}{m_\rho} \right\rfloor \right\rceil$$
$$< \frac{a_\rho}{\rho}\left(1 - \frac{1}{m_\rho}\right) + \frac{a_\rho}{m_\rho \lfloor \rho y \rfloor} + 2$$
$$= a_\rho\left(\frac{1}{\rho} - \frac{1}{\rho m_\rho} + \frac{1}{m_\rho \lfloor \rho y \rfloor}\right) + 2$$

$$= W_\rho + 2.$$

Summing all the inequalities for Case 2, we get

$$MH(L) < W(L) + 2k - 7. \quad \square$$

Lemma 2. Consider the weighting functions specified in Table I for $\alpha > 0$ and $\alpha = 0$. Let $y = \frac{265}{684}$, $k = 38$, $m_2 = 9$, $m_3 = 12$, and $m_\rho = \frac{(k-1)(\rho+1)}{k-\rho-1} = \frac{37(\rho+1)}{37-\rho}$ for $6 \le \rho \le k-2$ (see Table II). Then

$$W(L) < \left(\frac{3}{2} + \frac{1}{9} + \frac{1}{222}\right) OPT(L).$$

Proof. Consider a bin B in the optimal packing, and let $s_1 \ge s_2 \ge \cdots \ge s_t$ be the pieces packed in it, $s_1 + s_2 + \cdots + s_t \le 1$. Let $w(s_i)$ be the weight of a piece s_i, and let $w(B)$ be the weight of B. It is clear that

$$w(B) = w(s_1) + w(s_2) + \cdots + w(s_t).$$

We shall prove that

$$w(B) < \frac{3}{2} + \frac{1}{9} + \frac{1}{222},$$

both for $\alpha > 0$ and $\alpha = 0$.

The proof is done by cases, depending on the sizes of some of the largest pieces in B. Tables III and IV summarize the cases for $\alpha > 0$ and $\alpha = 0$, respectively. Columns s_1, s_2, and s_3 indicate the intervals containing the first, second, and third largest pieces in B. For convenience, we let I_j $(j \ge r)$ denote one of the intervals $I_r, I_{r+1}, \ldots, I_k$, or the "interval" [0]; i.e., no piece at all. Column $\bar{s}$ gives an upper bound on the remaining space left in the bin after packing the pieces of sizes specified in columns s_1, s_2, and s_3.

To determine an upper bound on $w(B)$, we find it useful to compute upper bounds on $\frac{w(s)}{s}$ for each possible piece size s (see Table II). In particular, we shall make use of the fact that

$$\sum_{j=i}^{t} w(s_j) \le \sum_{j=i}^{t} s_j \left(\max_{i \le r \le t} \frac{w(s_r)}{s_r}\right).$$

Interval containing piece s	$\alpha > 0$: $w(s)$	$\alpha > 0$: $\frac{w(s)}{s}$ (upper bound)	$\alpha = 0$: $w(s)$	$\alpha = 0$: $\frac{w(s)}{s}$ (upper bound)
$\bar{I}_1 : (1 - y, 1]$	1	$\frac{684}{419}$	1	$\frac{684}{419}$
$I_1 : (1/2, 1 - y]$	1	2	0	0
$\bar{I}_2 : (y, 1/2]$	$\frac{1}{2}$	$\frac{342}{265}$	$\frac{1}{2}$	$\frac{342}{265}$
$I_2 : (1/3, y]$	$\frac{4}{9}$	$\frac{4}{3}$	$\frac{5}{9}$	$\frac{5}{3}$
$I_3 : (1/4, 1/3]$	$\frac{11}{36}$	$\frac{11}{9}$	$\frac{7}{18}$	$\frac{14}{9}$
$I_4 : (1/5, 1/4]$	$\frac{1}{4}$	$\frac{5}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
$I_5 : (1/6, 1/5]$	$\frac{1}{5}$	$\frac{6}{5}$	$\frac{1}{5}$	$\frac{6}{5}$
$I_6 : (1/7, 1/6]$	$\frac{38}{7 * 37}$	$\frac{38}{37}$	$\frac{107}{14 * 37}$	$\frac{107}{2 * 37}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$I_{12} : (1/13, 1/12]$	$\frac{38}{13 * 37}$	$\frac{38}{37}$	$\frac{177}{52 * 37}$	$\frac{177}{4 * 37}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$I_{18} : (1/19, 1/18]$	$\frac{38}{19 * 37}$	$\frac{38}{37}$	$\frac{13}{6 * 37}$	$\frac{247}{6 * 37}$
$I_{19} : (1/20, 1/19]$	$\frac{38}{20 * 37}$	$\frac{38}{37}$	$\frac{71}{35 * 37}$	$\frac{284}{7 * 37}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$I_{k-2} : (1/(k-1), 1/(k-2)]$	$\frac{38}{37 * 37}$	$\frac{38}{37}$	$\frac{495}{481 * 37}$	$\frac{495}{481}$
$I_{k-1} : (1/k, 1/(k-1)]$	$\frac{1}{37}$	$\frac{38}{37}$	$\frac{1}{37}$	$\frac{38}{37}$
$I_k : (0, 1/k]$	$\frac{38}{37} s$	$\frac{38}{37}$	$\frac{38}{37} s$	$\frac{38}{37}$

Note: For $\alpha = 0$, $s \in I_j$, and $s' \in I_{j'}$, $6 \leq j \leq j' \leq k$, upper bound on $\frac{w(s)}{s} \geq$ upper bound on $\frac{w(s')}{s'}$.

Table II. Weight summary for all pieces s, when $y = \frac{265}{684}$, $k = 38$, $m_2 = 9$, $m_3 = 12$, and

$$m_\rho = \frac{(k-1)(\rho+1)}{k-\rho-1} = \frac{37(\rho+1)}{37-\rho} \text{ for } 6 \leq \rho \leq k-2.$$

s_1	s_2	s_3	$\bar{s}$	$w(B)$ $w(s_1) + w(s_2) + \cdots + w(s_t)$ (upper bound)
I_1	$\bar{I}_2$	-	$\frac{1}{2} - y = \frac{77}{684} < \frac{1}{6}$	$1 + \frac{1}{2} + \frac{77}{684} \cdot \frac{38}{37}$
I_1 or $\bar{I}_1$	I_2	-	$\frac{1}{6}$	$1 + \frac{4}{9} + \frac{1}{6} \cdot \frac{38}{37}$
I_1 or $\bar{I}_1$	I_3	I_4	$\frac{1}{20}$	$1 + \frac{11}{36} + \frac{1}{4} + \frac{1}{20} \cdot \frac{38}{37}$
I_1 or $\bar{I}_1$	I_3	I_j $(j \geq 5)$	$\frac{1}{4}$	$1 + \frac{11}{36} + \frac{1}{4} \cdot \frac{6}{5}$
I_1 or $\bar{I}_1$	I_4	I_4	$\frac{1}{10}$	$1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{10} \cdot \frac{38}{37}$
I_1 or $\bar{I}_1$	I_4	I_j $(j \geq 5)$	$\frac{3}{10}$	$1 + \frac{1}{4} + \frac{3}{10} \cdot \frac{6}{5}$
I_1 or $\bar{I}_1$	I_j $(j \geq 5)$	-	$\frac{1}{2}$	$1 + \frac{1}{2} \cdot \frac{6}{5}$
$\bar{I}_2$ or I_j $(j \geq 2)$	-	-	1	$1 \cdot \frac{4}{3}$

Table III. List of cases proving $w(B) < \frac{3}{2} + \frac{1}{9} + \frac{1}{222}$ for $\alpha > 0$.

s_1	s_2	s_3	$\bar{s}$	$w(B)$ $w(s_1) + w(s_2) + \cdots + w(s_t)$ (upper bound)
T_1	I_2	I_{18}	$y - \frac{1}{3} - \frac{1}{19} = \frac{1}{684}$	$1 + \frac{5}{9} + \frac{13}{6*37} + \frac{1}{684} * \frac{38}{37}$
T_1	I_2	$I_j\ (j \geq 19)$	$y - \frac{1}{3} = \frac{37}{684}$	$1 + \frac{5}{9} + \frac{37}{684} * \frac{284}{7*37}$
T_1	$I_j\ (j \geq 3)$	-	$y = \frac{265}{684}$	$1 + \frac{265}{684} * \frac{14}{9}$
I_1	-	-	$\frac{1}{2}$	$\frac{1}{2} * \frac{5}{3}$
T_2	T_2	-	$1 - 2y = \frac{154}{684} < \frac{1}{4}$	$\frac{1}{2} + \frac{1}{2} + \frac{154}{684} * \frac{107}{2*37}$
T_2 or I_2	I_2	I_3	$\frac{1}{12}$	$\frac{5}{9} + \frac{5}{9} + \frac{7}{18} + \frac{1}{12} * \frac{177}{4*37}$
T_2 or I_2	I_2	$I_j\ (j \geq 4)$	$\frac{1}{3}$	$\frac{5}{9} + \frac{5}{9} + \frac{1}{3} * \frac{107}{2*37}$
T_2 or I_2	$I_j\ (j \geq 3)$	-	$\frac{2}{3}$	$\frac{5}{9} + \frac{2}{3} * \frac{14}{9}$
$I_j\ (j \geq 3)$	-	-	1	$1 * \frac{14}{9}$

Table IV. List of cases proving $w(B) < \frac{3}{2} + \frac{1}{9} + \frac{1}{222}$ for $\alpha = 0$.

Column $w(B)$ in tables III and IV specifies an upper bound on $w(s_1) + w(s_2) + \cdots + w(s_t)$, calculated by taking $w(s_1)$ if s_1 is specified, plus $w(s_2)$ if s_2 is specified, plus $w(s_3)$ if s_3 is specified, plus $\bar{s} \max\limits_{s \leq \bar{s}} \frac{w(s)}{s}$.

For example, consider the case for $\alpha > 0$, with $s_1 \in I_1$ and $s_2 \in \bar{I}_2$. Then

$$\bar{r} = 1 - (s_1 + s_2) < 1 - (1/2 + y) = 1/2 - y = \frac{77}{684} < 1/6.$$

Thus, we know that each of $s_3, \ldots, s_t$ must be smaller than 1/6, and so (from Table II) $\frac{w(s_i)}{s_i} \leq \frac{38}{37}$ for $3 \leq i \leq t$. This gives

$$\begin{aligned} w(B) &= w(s_1) + w(s_2) + \sum_{i=3}^{t} w(s_i) \\ &< 1 + \frac{1}{2} + \frac{77}{684} * \frac{38}{37} \\ &= \frac{3}{2} + \frac{1}{9} + \frac{1}{222}. \end{aligned}$$

As another example, consider the case for $\alpha > 0$, with $s_1 \in I_1$ and $s_2 \in I_j$ ($j \geq 5$). Then $s_1 > 1/2$, and $\bar{r} = 1 - s_1 < 1/2$. The $\bar{r}$ portion of the bin has no piece larger than 1/5, and so $\frac{w(s)}{s} \leq \frac{6}{5}$ for any piece in B other than s_1. This gives

$$w(B) = w(s_1) + \sum_{i=2}^{t} w(s_i) < 1 + \frac{1}{2} * \frac{6}{5} < \frac{3}{2} + \frac{1}{9}.$$

As another example, consider the case for $\alpha = 0$, with $s_1 \in I_2$, $s_2 \in I_2$, and $s_3 \in I_3$. Then

$$\bar{r} = 1 - (s_1 + s_2 + s_3) < 1 - (1/3 + 1/3 + 1/4) = 1/12.$$

Thus, we know that each of $s_4, \ldots, s_t$ must be smaller than 1/12, and so $\frac{w(s_i)}{s_i} \leq \frac{177}{4 * 37}$ for $4 \leq i \leq t$. This gives

$$\begin{aligned} w(B) &= w(s_1) + w(s_2) + w(s_3) + \sum_{i=4}^{t} w(s_i) \\ &< \frac{5}{9} + \frac{5}{9} + \frac{7}{18} + \frac{1}{12} * \frac{177}{4 * 37} \\ &< \frac{3}{2} + \frac{1}{9}. \end{aligned}$$

Tables III and IV enumerate all possible cases, and it is easily checked that the values in the $w(B)$ column are all at most $\frac{3}{2} + \frac{1}{9} + \frac{1}{222}$. Thus, the result of the lemma follows. □

Lemmas 1 and 2 guarantee that the performance ratio of MH is less than $\frac{3}{2} + \frac{1}{9} + \frac{1}{222}$. We now show that this bound is essentially tight.

Lemma 3. Let $y = \frac{265}{684}$, $k = 38$, $m_2 = 9$, $m_3 = 12$, and $m_\rho = \frac{(k-1)(\rho+1)}{k-\rho-1} = \frac{37(\rho+1)}{37-\rho}$ for $6 \le \rho \le k-2$. Then

$$R_{MH} \geq \frac{3}{2} + \frac{1}{9} + \frac{1}{222} - \frac{1}{987{,}012}.$$

Proof. We shall exhibit lists L with arbitrarily large $OPT(L)$ such that

$$\frac{MH(L)}{OPT(L)} = \frac{3}{2} + \frac{1}{9} + \frac{1}{222} - \frac{1}{987{,}012}.$$

Let n be a multiple of 24,675,300, and let

$$L_1 = (p_1, p_2, \ldots, p_n),$$

$$L_2 = (q_1, q_2, \ldots, q_n),$$

$$L_3 = (t_1, t_2, \ldots, t_n),$$

$$L_4 = (u_1, u_2, \ldots, u_{2n}),$$

and

$$L_5 = (v_1, v_2, \ldots, v_n),$$

where

$$p_i = \frac{1}{2} + \epsilon,$$

$$q_i = y + \epsilon,$$

$$t_i = \frac{1}{26} + \epsilon,$$

$$u_i = \frac{1}{27} + \epsilon,$$

and

$$v_i = \frac{1}{26{,}676} - 5\epsilon,$$

for some ϵ, $0 < \epsilon < 10^{-10}$. Let L be the list obtained by concatenating these lists; i.e., $L = L_1L_2L_3L_4L_5$.

Note that $p_i + q_i + t_i + u_i + u_j + v_i = 1$. Hence, the packing in which each bin contains a p_i piece, a q_i piece, a t_i piece, two u_i pieces, and a v_i piece is an optimal packing, and so $OPT(L) = n$.

Now consider the packing produced by our algorithm. Note that $m_{25} = \frac{481}{6}$ and $m_{26} = \frac{999}{11}$. Since $\lfloor 25y \rfloor = 9$ and $\lfloor 26y \rfloor = 10$, 9 t_i pieces or 10 u_i pieces can be packed in the same bin with a p_i piece. Thus, the packing produced by our algorithm consists of

$n - \frac{n}{9m_{25}} - \frac{2n}{10m_{26}} = \frac{64{,}702\,n}{64{,}935}$ bins, each containing 1 p_i piece,

$\frac{n}{9m_{25}} = \frac{2n}{1{,}443}$ bins, each containing 1 p_i piece, and 9 t_i pieces,

$\frac{2n}{10m_{26}} = \frac{11n}{4{,}995}$ bins, each containing 1 p_i piece, and 10 u_i pieces,

$\frac{n}{25}\left(1 - \frac{1}{m_{25}}\right) = \frac{19n}{481}$ bins, each containing 25 t_i pieces,

$\frac{2n}{26}\left(1 - \frac{1}{m_{26}}\right) = \frac{76n}{999}$ bins, each containing 26 u_i pieces,

$\frac{n}{2}$ bins, each containing 2 q_i pieces,

and

$\frac{n}{26{,}676}$ bins, each containing 26,676 v_i pieces.

Summing, we obtain

$$MH(L) = \left(\frac{3}{2} + \frac{1}{9} + \frac{1}{222} - \frac{1}{987{,}012}\right)n,$$

or

$$\frac{MH(L)}{OPT(L)} = \frac{3}{2} + \frac{1}{9} + \frac{1}{222} - \frac{1}{987{,}012}.$$

Since $OPT(L) = n$ can be arbitrarily large, the result follows. □

Note that $987{,}012 = 26{,}676 * 37$. The discrepancy of $1/987{,}012$ in our analysis is due to the weights we assigned to the I_k-pieces. For the list considered in the proof of Lemma 3, bins of type B_k are completely packed, instead of being only $\frac{k-1}{k}$ full.

Lemmas 1, 2, and 3 are summarized by the following theorem.

Theorem 1. Let $y = \frac{265}{684}$, $k = 38$, $m_2 = 9$, $m_3 = 12$, and $m_\rho = \frac{(k-1)(\rho+1)}{k-\rho-1} = \frac{37(\rho+1)}{37-\rho}$ for $6 \le \rho \le k-2$. Then

$$\frac{3}{2} + \frac{1}{9} + \frac{1}{222} - \frac{1}{987{,}012} \le R_{MH} < \frac{3}{2} + \frac{1}{9} + \frac{1}{222};$$

i.e.,

$$1.6156146 < R_{MH} < 1.61(561)^{*}.$$

4. A General Lower Bound

In this section, we present a lower bound for a class of algorithms, which includes the Modified Harmonic Algorithm presented in Section 2.

Let C be the class of algorithms which behave as follows. If $A \in C$, then A divides the interval $(0, 1]$ into disjoint subintervals, including $\bar{I}_1 = (1-y, 1]$, $I_1 = (1/2, 1-y]$, $\bar{I}_2 = (y, 1/2]$, $I_2 = (1/3, y]$, and $I_\lambda = (0, \lambda]$, for some y and λ, $1/3 < y < 1/2$ and $0 < \lambda \le 1/3$. Pieces are classified according to the intervals to which they belong. The packing produced by A must obey the following rules:

($R1$) The number of bins which contain only one I_2-piece (with or without pieces of other types) is a fixed fraction $1/m$ (m need not be an integer, and if this fraction is zero, we take m to be ∞) of the total number of I_2-pieces in the input list.

($R2$) No bin may contain

(i) an $\bar{I}_1$-piece and an I_2-piece,

or

(ii) an I_1-piece and an $\bar{I}_2$-piece.

($R3$) No bin may contain an I_λ-piece together with an $\bar{I}_1$-piece, I_1-piece, $\bar{I}_2$-piece, or an I_2-piece.

Note that the Modified Harmonic Algorithm is in C. Also note that the above rule $R2$ rules out packing $I_1 \cup \bar{I}_1$-pieces with I_2-pieces, or I_1-pieces with $I_2 \cup \bar{I}_2$-pieces, by First-Fit ($O(n \lg n)$ time). We have the following lower bound on the performance of any algorithm in C.

Theorem 2. For any algorithm $A \in C$, $R_A \geq \frac{3}{2} + \frac{1}{9} = 1.61^*$.

Proof. We shall exhibit three kinds of lists L with arbitrarily large $OPT(L)$, and show that the average value of $\frac{A(L)}{OPT(L)}$ for these three kinds of lists is at least $\frac{3}{2} + \frac{1}{9}$. Each of these three kinds of lists will be of the form $L = L_1 L_2 L_3$, where $L_1 = (u_1, u_2, \ldots, u_n)$, $L_2 = (v_1, v_2, \ldots, v_n)$, and $L_3 = (t_1, t_2, \ldots, t_{Mn})$. Moreover, we will always have $u_i + v_i + \sum_{j=1}^{M} t_{i_j} = 1$. Hence, the packing in which each bin contains a u_i piece, a v_i piece, and M t_i pieces is an optimal packing, and so $OPT(L) = n$.

Throughout the proof, we let n be a positive integer that is a multiple of $6m$ (for simplicity, we assume that m is rational), $M = \left\lceil \frac{1}{6\lambda} \right\rceil$, and ϵ be such that $0 < \epsilon < \min \left\{ \frac{(1/2 - y)^2}{(M+1)^2}, \frac{(y - 1/3)^2}{(M+1)^2} \right\}$.

Instance 1. Let

$$u_i = \frac{1}{2} + \frac{M}{2}\epsilon,$$

$$v_i = y + \frac{M}{2}\epsilon,$$

and

$$t_i = \frac{1/2 - y}{M} - \epsilon.$$

Clearly, $u_i \in I_1$, $v_i \in \overline{I}_2$, and $t_i \in I_\lambda$.

Now consider the packing produced by algorithm A. By rules $R1$ to $R3$ imposed on the algorithms in class C, this packing consists of

n bins, each containing 1 u_i piece,

at least $\frac{n}{2}$ bins containing v_i pieces,

and

at least $\left\lceil \dfrac{nM}{\left\lfloor \dfrac{M}{1/2 - y} \right\rfloor} \right\rceil$ bins containing t_i pieces.

Hence

$$A(L) \geq n + \frac{n}{2} + \left\lceil \frac{nM}{\left\lfloor \frac{M}{1/2 - y} \right\rfloor} \right\rceil \geq (2 - y)n.$$

Since $OPT(L) = n$ can be arbitrarily large, we have

$$R_A \geq 2 - y. \tag{1}$$

Instance 2. Let

$$u_i = \frac{1}{2} + \frac{M}{2}\epsilon,$$

$$v_i = \frac{1}{3} + \frac{M}{2}\epsilon,$$

and

$$t_i = \frac{1}{6M} - \epsilon.$$

Clearly, $u_i \in I_1$, $v_i \in I_2$, and $t_i \in I_\lambda$.

Now consider the packing produced by algorithm A. This packing consists of

$\frac{n}{2}\left(1 - \frac{1}{m}\right)$ bins, each containing 2 v_i pieces,

$\frac{n}{m}$ bins, each containing 1 v_i piece (and may be 1 u_i piece),

at least $n\left(1 - \frac{1}{m}\right)$ bins, each containing 1 u_i piece alone,

and

at least $\frac{n}{6}$ bins containing t_i pieces.

Hence

$$A(L) \geq \frac{n}{2}\left(1 - \frac{1}{m}\right) + \frac{n}{m} + n\left(1 - \frac{1}{m}\right) + \frac{n}{6} = \left(\frac{5}{3} - \frac{1}{2m}\right)n.$$

Since $OPT(L) = n$ can be arbitrarily large, we have

$$R_A \geq \frac{5}{3} - \frac{1}{2m}. \tag{2}$$

Instance 3. Let

$$u_i = 1 - y + \frac{M}{2}\epsilon,$$

$$v_i = \frac{1}{3} + \frac{M}{2}\epsilon,$$

and

$$t_i = \frac{y - 1/3}{M} - \epsilon.$$

Clearly, $u_i \in \overline{I}_1$, $v_i \in I_2$, and $t_i \in I_\lambda$.

Now consider the packing produced by algorithm A. This packing consists of

n bins, each containing 1 u_i piece,

$\frac{n}{2}\left(1 - \frac{1}{m}\right)$ bins, each containing 2 v_i pieces,

$\frac{n}{m}$ bins, each containing 1 v_i piece,

and

at least $\left\lceil \frac{nM}{\left\lfloor \frac{M}{y - 1/3} \right\rfloor} \right\rceil$ bins containing t_i pieces.

Hence

$$A(L) \geq n + \frac{n}{2}\left(1 - \frac{1}{m}\right) + \frac{n}{m} + \left\lceil \frac{nM}{\left\lfloor \frac{M}{y - 1/3} \right\rfloor} \right\rceil \geq \left(\frac{7}{6} + y + \frac{1}{2m}\right)n.$$

Since $OPT(L) = n$ can be arbitrarily large, we have

$$R_A \geq \frac{7}{6} + y + \frac{1}{2m}. \tag{3}$$

Adding (1), (2), and (3), we have

$$3R_A \geq \frac{29}{6},$$

or

$$R_A \geq \frac{3}{2} + \frac{1}{9}. \quad \square$$

If the algorithm A in Theorem 2 uses Next-Fit to pack I_λ-pieces, then we can get the improved result $R_A > \frac{3}{2} + \frac{1}{9}$, by modifying the instances 1, 2, and 3 in the proof as follows. In L_3 introduce pieces whose sizes are of the form $k\epsilon$ for some appropriate positive integer k, at regular frequency. For example, in Instance 2, every $6M - 1$ t_i pieces will be followed by a piece of size $(6M + 1)\epsilon$. Since ϵ can be taken to be arbitrarily small, we have $OPT(L) = n + 1$. But $A(L) \geq \left(\frac{3}{2} + \frac{M}{6(M-1)} - \frac{1}{2m}\right)n$.

5. Conclusions

We have presented a new on-line algorithm, the Modified Harmonic Algorithm, which has a better asymptotic performance ratio than any previously known on-line algorithm. Moreover, this algorithm is a linear-time algorithm. It seems likely that a better algorithm could be constructed if the linear time constraint were relaxed.

It is argued in [LL83] that any linear-time, constant-space on-line algorithm has a performance ratio of at least 1.692... . We relaxed the constant space constraint and achieved 1.61(561)*. In the previous section we described a class of linear-time on-line algorithms which have a performance ratio of at least 1.61*. It seems quite likely that no linear-time on-line algorithm can do better.

Finally, we observe that our algorithm leads to improved on-line algorithms for packing in two-dimensions. For packing rectangles in a strip, we can devise a shelf algorithm similar to those in [BS83]. By choosing appropriate shelf heights, we can obtain a performance ratio arbitrarily close to R_{MH}. We can also devise an on-line algorithm for the problem of packing rectangles in finite two-dimensional bins discussed in [CGJ82]. We believe that our on-line algorithm for this problem will have a performance ratio R_{MH}^2. These will be discussed in more detail in [R84].

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